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Spontaneous magnetisations of the Ising model on the bathroom tile lattice

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Abstract. We evaluate the site spontaneous magnetisations on the general 4-8 or bathroom tile lattice using mappings to checkerboard, free-fermion and Union Jack Ising models. Our results verify the conjecture of Lin *et al*, who proposed their formula from twelfth-order low-temperature series expansions.

1. Introduction

In two recent papers, Lin and Fang (1985) and Lin *et al* (1987) considered the spontaneous magnetisation of the exactly solvable two-dimensional Ising model on a 4-8 or bathroom tile lattice (figure 1), whose partition function was given by Utiyama (1951) over thirty years ago. Lin *et al* (1987) corrected the earlier results of Lin and Fang (1985) for the spontaneous magnetisation $\langle\sigma_i\rangle$, and proposed a conjecture for $\langle\sigma_i\rangle$ after investigating a twelfth-order low-temperature series expansion. They originally used the traditional route of Onsager (1949) and Montroll *et al* (1963) and found an apparent impassé in evaluating limits of block Toeplitz determinants (Lin *et al* 1987). In this paper we provide an exact derivation of the site spontaneous magnetisations using mappings onto free-fermion and checkerboard Ising models (Baxter 1986). Such mappings were used previously to obtain the site spontaneous magnetisations of the related (dual) Union Jack lattice (Choy and Baxter 1987). The basic approach here is not to follow the traditional route of Onsager (1949) and Montroll *et al* (1963) which involves evaluation of Toeplitz determinants but rather via mappings to known results of the free-fermion and checkerboard Ising models (Baxter 1986). Using these transformations we first derive an identity relating long-distance correlations on the bathroom tile lattice to those of the checkerboard Ising model. By taking the infinite-distance limit we then relate the spontaneous magnetisations $\langle\sigma_1\rangle$ and $\langle\sigma_2\rangle$ (figure 1) to M_0, the spontaneous magnetisation of the checkerboard model (Baxter 1986). Thus we divide this paper into three main sections. Section 2 discusses various mappings of the bathroom tile with its dual, the Union Jack lattice which is related to the checkerboard and free-fermion models. Section 3 derives the correlation identity mentioned above by modifications of these models and reversing the mappings. Section 4 provides the solution of the two spontaneous magnetisations $\langle\sigma_1\rangle$ and $\langle\sigma_2\rangle$ for the general bathroom tile lattice (figure 1). The other two, $\langle\sigma_3\rangle$ and $\langle\sigma_4\rangle$, follow from symmetry and a trivial permutation of labels. Agreement with the conjectured formula for the special case of Lin *et al* (1987) will also be indicated here. We follow the notation of Baxter (1986) for consistency wherever possible, which differs from the notation of Lin *et al* (1987).

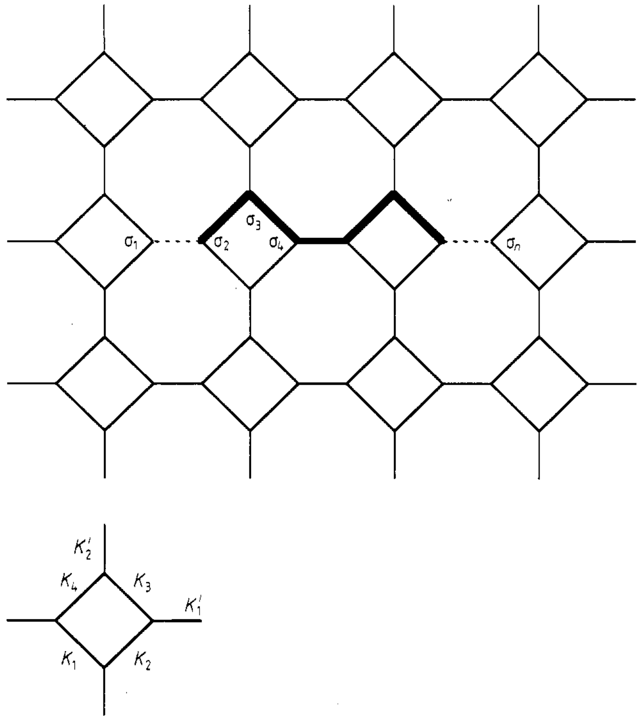

Figure 1. Two-dimensional Ising model on a bathroom tile lattice (model 1) with six coupling constants K_1, K_2, K_3, K_4, K'_1 and K'_2. Bold lines have extra factors $\exp(K_i\sigma_i\sigma_j)\rightarrow \sigma_i\sigma_j\exp(K_i\sigma_i\sigma_j)$ with the exception of the left and right pairs (broken lines) as discussed later.

2. Mappings

Hereafter we shall refer to the Ising model on the bathroom tile lattice of figure 1 as model 1. We start our route outwards by the duality transformation (Kramers and Wannier 1941; see also Baxter 1982) that the dual of model 1 is a Union Jack lattice which we refer to as model 2 (see figure 2) with the bonds given by:

$$\text{model 2}\qquad \begin{aligned}&\exp(-2L_i)=\tanh K_i \qquad i=1,2,3,4\\ &\exp(-2L'_j)=\tanh K'_j \qquad j=1,2.\end{aligned} \tag{1}$$

By summing over fourfold coordinated spins, this is related to a free-fermion model (Baxter 1986, Choy and Baxter 1987) (our model 3; see figure 3) with weights given by

$$\text{model 3}\qquad \begin{aligned}W(a,b,c,d)=2\exp\{[L'_1(ad+bc)\\ +L'_2(ab+cd)]/2\}\cosh(L_1a+L_2b+L_3c+L_4d)\end{aligned} \tag{2}$$

which we rewrite as follows:

$$\omega_1=W(++++)=2\exp(L'_1+L'_2)\cosh(L_1+L_2+L_3+L_4)=\rho(1+t_1t_2t_3t_4)$$

$$\omega_2=W(+-+-)=2\exp(-L'_1-L'_2)\cosh(L_1-L_2+L_3-L_4)=\rho t'_1t'_2(t_1t_3+t_2t_4)$$

$$\omega_3=W(+--+)=2\exp(L'_1-L'_2)\cosh(L_1-L_2-L_3+L_4)=\rho t'_2(t_1t_4+t_2t_3)$$

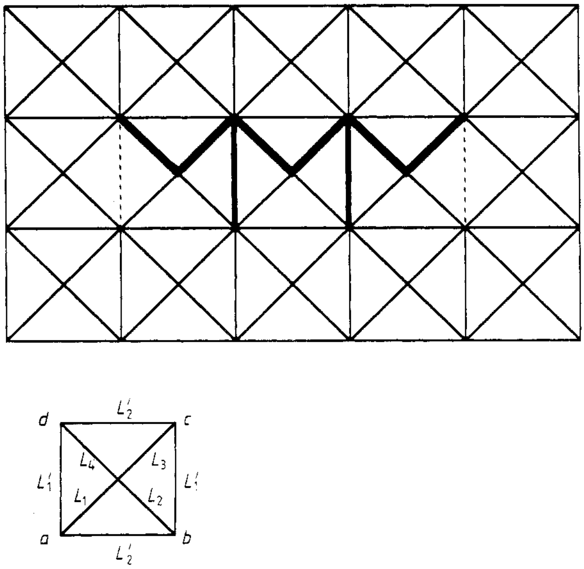

Figure 2. Union Jack lattice (model 2) related to model 1 via the duality transformation (1) and equivalent to the free-fermion model of figure 3. On bold lines the interaction coefficients (L_3, L_4, L_1') are negated. On the right (left) broken line the coefficient L_1' is replaced by $\mu(-\mu)$.

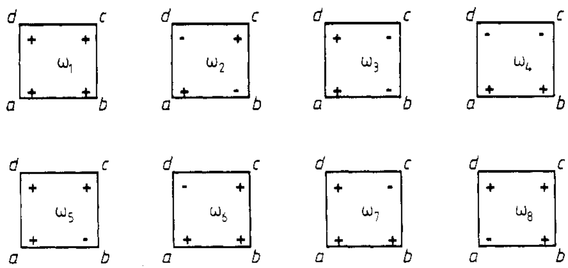

Figure 3. Free-fermion model related to the Union-Jack model 2 with weights $\omega_1, \omega_2, \ldots, \omega_8$ given by (3) and also equivalent to a checkerboard Ising model (model 5).

$$\begin{aligned}
\omega_4 &= W(++--) = 2\exp(-L_1'+L_2')\cosh(L_1+L_2-L_3-L_4) = \rho t_1'(t_1t_2+t_3t_4) \\
\omega_5 &= W(+-++) = 2\cosh(L_1-L_2+L_3+L_4) = \rho(t_2+t_1t_3t_4)(t_1't_2')^{1/2} \\
\omega_6 &= W(+++-) = 2\cosh(L_1+L_2+L_3-L_4) = \rho(t_4+t_1t_2t_3)(t_1't_2')^{1/2} \\
\omega_7 &= W(++-+) = 2\cosh(L_1+L_2-L_3+L_4) = \rho(t_3+t_1t_2t_4)(t_1't_2')^{1/2} \\
\omega_8 &= W(-+++) = 2\cosh(-L_1+L_2+L_3+L_4) = \rho(t_1+t_2t_3t_4)(t_1't_2')^{1/2}
\end{aligned} \tag{3}$$

where

$$\begin{aligned}
t_i &= \tanh K_i \qquad \text{for } i=1,2,3,4 \\
t_j' &= \tanh K_j' \qquad \text{for } j=1,2
\end{aligned} \tag{4}$$

and ρ is a normalisation factor. Unfortunately this model has $\omega_5 \neq \omega_6$ and $\omega_7 \neq \omega_8$ which is inconsistent with the derivations of Baxter (1986) (see equation (2.9) of this reference). To overcome this inconvenience we shall make a trivial modification of model 3 to model 4 which has weights:

model 4 $$W_F(a, b, c, d) = \exp\{-[X(cd-ab)+Y(ad-bc)]/2\}\,W(a, b, c, d) \tag{5}$$

where we choose X, Y such that

$$W_F(a, b, c, d) = W_F(c, d, a, b) \tag{6}$$

i.e.

$$\begin{aligned}\omega_{F5} &= e^{-X-Y}\omega_5 = \omega_{F6} = e^{X+Y}\omega_6\\ \omega_{F7} &= e^{X-Y}\omega_7 = \omega_{F8} = e^{Y-X}\omega_8\end{aligned} \tag{7}$$

so

$$\begin{aligned}e^{4X} &= \omega_5\omega_8/\omega_6\omega_7\\ e^{4Y} &= \omega_5\omega_7/\omega_6\omega_8.\end{aligned} \tag{8}$$

Then from Baxter (1986) there exists a checkerboard Ising model (our model 5; see figure 4) with ρ, M, P, J_1, J_2, J_3, J_4 such that

model 5 $$W_F(a, b, c, d) = 2\rho\exp\{[P(cd-ab)+M(ad-bc)]/2\}\cosh(J_1a+J_2b+J_3c+J_4d) \tag{9}$$

where solutions for ρ, M, P, J_1, J_2, J_3, J_4 are all given in terms of weights ω_1, ω_2, ω_3, ω_4, ω_5, ω_6, ω_7 and ω_8 (Baxter 1986). In particular we shall later require

$$\begin{aligned}\cosh 2P &= (\omega_1\omega_4+\omega_2\omega_3)/(2\sqrt{\omega_5\omega_6\omega_7\omega_8})\\ \cosh 2M &= (\omega_1\omega_3+\omega_2\omega_4)/(2\sqrt{\omega_5\omega_6\omega_7\omega_8}).\end{aligned} \tag{10}$$

These two expressions, and our later equations (22) and (23), are invariant under the above transformation $\omega_5 \Rightarrow \omega_{F5}$, $\omega_6 \Rightarrow \omega_{F6}$, $\omega_7 \Rightarrow \omega_{F7}$, $\omega_8 \Rightarrow \omega_{F8}$. From (5) it follows that

$$W(a, b, c, d) = 2\rho\exp\{[\lambda(cd-ab)+\mu(ad-bc)]/2\}\cosh(J_1a+J_2b+J_3c+J_4d) \tag{11}$$

where $\lambda = X+P$ and $\mu = Y+M$ with X, Y given by (8), and M, P by (10).

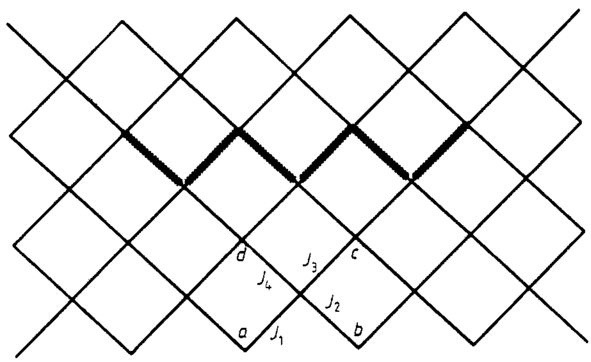

Figure 4. Checkerboard Ising model equivalent to the free-fermion model of figure 3. Edges with hatched lines have their interaction coefficients J_i replaced by $-J_i$ in the modified dual partition function Z^D_{mod} of (14).

Finally we define model 6 via a duality transformation of model 5, i.e.

$$\text{model 6} \qquad \exp(-2\hat{J}_i) = \tanh J_i \qquad i = 1, 2, 3, 4 \tag{12}$$

which completes an exposition of all the mappings we shall use in the next section where the sequence is reversed.

3. Correlation identity

We start with model 6 and consider the two-spin correlation $\langle\sigma_1\sigma_n\rangle$ of figure 5 with the lattice drawn diagonally. Then the expression for $\langle\sigma_1\sigma_n\rangle$ is

$$\langle\sigma_1\sigma_n\rangle = \langle\sigma_1\ \sigma_2\sigma_2\ \sigma_3\sigma_3\sigma_3\sigma_4 \ldots \sigma_n\rangle = Z_{\text{mod}}(\hat{J}_i)/Z(\hat{J}_i) \tag{13}$$

where Z is the partition function and Z_{mod} is a modification of Z by converting the weights of some edges (bold lines in figure 5) from $\exp(\hat{J}_i\sigma_i\sigma_j)$ to $\sigma_i\sigma_j \exp(\hat{J}_i\sigma_i\sigma_j)$, leaving the rest of the Boltzmann factors unaltered in Z. Now we make a duality transformation (12) going to model 5. Then

$$\langle\sigma_1\sigma_n\rangle = Z^{\text{D}}_{\text{mod}}(J_i)/Z^{\text{D}}(J_i) \tag{14}$$

where to within cancelling normalisation factors, Z^{D} is the dual of Z and $Z^{\text{D}}_{\text{mod}}$ is the dual of Z_{mod}. From (12) the equivalent modification in this case is obtained by converting J_i to $-J_i$ along the hatched lines in figure 4, which are the dual of the bold lines in figure 5. Continuing backwards to model 4 we see that the effect of this negation of bonds is to modify shaded faces (figure 6) of the free-fermion model 4 so that their weights (11) become:

$$\begin{aligned} W(a, b, c, d) &\Rightarrow J_3,\ J_4 \text{ negated in (11)} \\ &\equiv W(a, b, -c, -d) \qquad \text{and } \mu \to -\mu. \end{aligned} \tag{15}$$

The λ and μ weights in (11) cancel between adjacent faces, except for the leftmost and the rightmost edges of the shaded faces (broken lines in figure 6) which acquire weights $\exp(-\mu ad)$ and $\exp(\mu b'c')$ respectively. Going on to the equivalent model 2, the Union Jack (we bypass model 3 which is a trivial modification of model 4; see § 2), we see that these negations have the effect of negating L_3, L_4 and L_1' along the

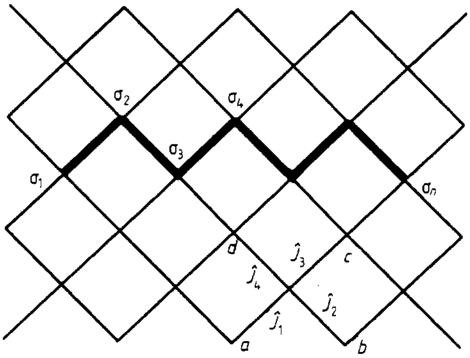

Figure 5. Two-spin correlation $\langle\sigma_1\sigma_n\rangle$ on the dual checkerboard Ising model (model 6). Edges (i, j) with bold lines have an extra factor $\sigma_i\sigma_j$ in the modified partition function, Z_{mod} of (13).

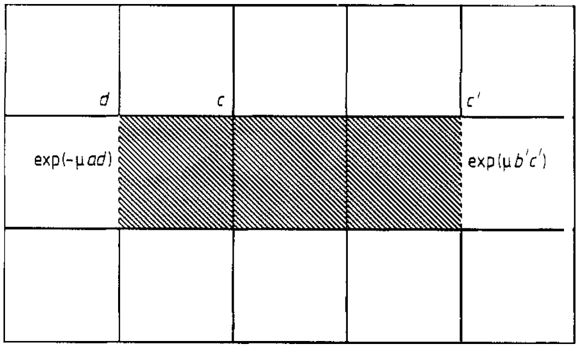

Figure 6. Free-fermion model equivalent of model 5. The shaded regions have weights $W(a, b, c, d)$ modified to $W(a, b, -c, -d)$ and $\mu \to -\mu$. The broken lines have exceptional boundary weights with additional factors shown.

heavy lines of figure 2, with the exception for the leftmost and rightmost edges which are replaced by $-\mu$ and μ respectively. Now we make the final step in the journey back to model 1, our bathroom tile, via a duality transformation. By similar arguments as for the step from model 6 to model 5 we see the effect of negating bonds on the Union Jack as introducing extra factors $\exp(K_i\sigma_i\sigma_j) \Rightarrow \sigma_i\sigma_j \exp(K_i\sigma_i\sigma_j)$ along the path (bold lines) in figure 1 as indicated. There are extra factors on the leftmost and rightmost boundary pairs that must now be considered which include normalisation factors R and S respectively, in the duality transformation. For the leftmost pair this is given by

$$(\text{dual of } \exp(-\mu ad))/(\text{dual of } \exp(L_1'ad)) = (R/S)\exp[(\hat{\mu} - K_1')ad] \tag{16}$$

and for the rightmost pair by

$$(\text{dual of } \exp(\mu b'c'))/(\text{dual of } \exp(L_1'b'c')) = (R/S)b'c'\exp[(\hat{\mu} - K_1')b'c'] \tag{17}$$

where $R\exp\hat{\mu} = \cosh\mu$, $R\exp(-\hat{\mu}) = \sinh\mu$, $S\exp K_1' = \cosh L_1'$ and $S\exp(-K_1') = \sinh L_1'$, which follows from duality. Thus S, R and $\hat{\mu}$ are completely determined by L_1' and μ; in particular

$$\begin{aligned}(R/S)\exp(\hat{\mu} - K_1') &= \cosh\mu/\cosh L_1' \\ (R/S)\exp(K_1' - \hat{\mu}) &= \sinh\mu/\sinh L_1'.\end{aligned} \tag{18}$$

From these results our correlation identity follows:

$$\begin{aligned}&(R^2/S^2)\langle[\cosh(\hat{\mu} - K_1') + \sigma_1\sigma_2\sinh(\hat{\mu} - K_1')]\sigma_2\sigma_3\sigma_3\sigma_4\ldots \\ &\qquad\times\sigma_{n-1}\sigma_{n-1}\sigma_n[\cosh(\hat{\mu} - K_1') + \sigma_{n-1}\sigma_n\sinh(\hat{\mu} - K_1')]\rangle_{\text{model 1}} \\ &\quad= \langle\sigma_1\sigma_n\rangle_{\text{model 6}}\end{aligned} \tag{19}$$

which relates correlations on the bathroom tile lattice (model 1) to those of the checkerboard Ising model (model 6). Since $\sigma_i^2 = 1$, the intermediate σ_i cancel, leaving

$$\begin{aligned}&(R^2/S^2)\langle[\cosh(\hat{\mu} - K_1')\sigma_2 + \sigma_1\sinh(\hat{\mu} - K_1')] \\ &\qquad\times[\cosh(\hat{\mu} - K_1')\sigma_n + \sigma_{n-1}\sinh(\hat{\mu} - K_1')]\rangle_{\text{model 1}} \\ &\quad= \langle\sigma_1\sigma_n\rangle_{\text{model 6}}.\end{aligned} \tag{20}$$

4. Spontaneous magnetisations

Consider the infinite distance ($n \to \infty$) limit of (19). We obtain

$$M_0^2 = (R^2/S^2)[\cosh(\hat{\mu} - K_1')\langle\sigma_2\rangle + \langle\sigma_1\rangle \sinh(\hat{\mu} - K_1')]^2 \tag{21}$$

where M_0 is the spontaneous magnetisation of the dual checkerboard model 6. This is in turn the disorder parameter of model 5 and of the free-fermion models 3 and 4. We define Γ_2, h_2, Ω as in Baxter (1986):

$$\Gamma_2 = \operatorname{sech} 2M = \frac{2(\omega_5\omega_6\omega_7\omega_8)^{1/2}}{\omega_1\omega_3 + \omega_2\omega_4} \tag{22}$$

$$h_2 = (\omega_2^2 + \omega_4^4 - \omega_1^2 - \omega_3^2)/[2(\omega_1\omega_3 + \omega_2\omega_4)] \tag{23}$$

$$\Omega^2 = (\Gamma_2^2 + h_2^2 - 1)/\Gamma_2^2. \tag{24}$$

M_0 is obtained from equation (4.19) of Baxter (1986), except that Ω is inverted because our M_0 is the spontaneous magnetisation of the dual model. Thus

$$M_0 = (1 - \Omega^2)^{1/8} = \left(\frac{(s-\omega_1)(s-\omega_2)(s-\omega_3)(s-\omega_4)}{\omega_5\omega_6\omega_7\omega_8}\right)^{1/8} \tag{25}$$

where $s = (\omega_1 + \omega_2 + \omega_3 + \omega_4)/2$.

Taking the square root of (21), using (18) and the fact that $\mu = Y + M$, we obtain

$$F\cosh(Y+M) + G\sinh(Y+M) = M_0 \tag{26}$$

where

$$2F = (\langle\sigma_2\rangle + \langle\sigma_1\rangle)/\cosh L_1'$$
$$2G = (\langle\sigma_2\rangle + \langle\sigma_1\rangle)/\sinh L_1'. \tag{27}$$

If the bathroom tile model parameters $K_1, \ldots, K_4$, K_1', K_2' are given, then $L_1, \ldots, L_4$, L_1', L_2' are defined by (1) and $\omega_1, \ldots, \omega_8$ by (3); Y by (7) and M by (10). There are actually two solutions for M: $+M$ and $-M$, just as there are two solutions for P ($+P$ and $-P$), corresponding to the fact that to a given free-fermion model there correspond four equivalent checkerboard Ising models (Baxter 1986).

We could have used any one of these four checkerboard models as our model 5: the choice cannot affect M_0, $\langle\sigma_1\rangle$ or $\langle\sigma_2\rangle$; so in addition to (26) we have

$$F\cosh(Y-M) + G\sinh(Y-M) = M_0. \tag{28}$$

We can solve (26) and (28) for F and G:

$$F = M_0 \cosh Y/\cosh M$$
$$G = -M_0 \sinh Y/\cosh M \tag{29}$$

and (27) then yields

$$\langle\sigma_1\rangle = M_0 \cosh(L_1' + Y)/\cosh M$$
$$\langle\sigma_2\rangle = M_0 \cosh(L_1' - Y)/\cosh M. \tag{30}$$

From symmmetry $\langle\sigma_3\rangle$ and $\langle\sigma_4\rangle$ are given by changing $L_1' \to L_2'$, $M \to P$ and $Y \to -X$ in (30) respectively. For the special case of $K_1 = K_3$, $K_2 = K_4$ of Lin *et al* (1987) we find

$$\langle\sigma_1\rangle = \langle\sigma_2\rangle = M_0\left(1 + \frac{4yxx'}{(1+x^2)(1+x'^2)}\right)^{-1/2} \tag{31}$$

where $y = \exp(-2K_1')$, $x = \exp(-2K_2)$ and $x' = \exp(-K_1)$, in agreement with their conjecture.

5. Conclusion

In this paper we derived a correlation identity for the 2-spin correlation function between Ising models on the bathroom tile and the checkerboard lattices. By considering the infinite distance limit of this correlation identity we related the site spontaneous magnetisations of the bathroom tile lattice with that of the checkerboard model. We have thus proved the conjectured formula of Lin *et al* (1987) who have earlier proposed their formula from a twelfth-order low-temperature series expansion.

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